MIT OCW GR PSET 3

1 · Accelerated observer revisited

a · On a spacetime diagram, show the trajectory (t, x) exhibited by the uniformly accelerated astronaut from HW 2, problem 6:

· We previously found that the four-velocity of the astronaut is:

$$\vec{u} = (\cosh(g\bar{t}), \sinh(g\bar{t}), 0, 0)$$
$$= (u^0, u^1, u^2, u^3) = (u^t, u^x, u^y, u^z)$$

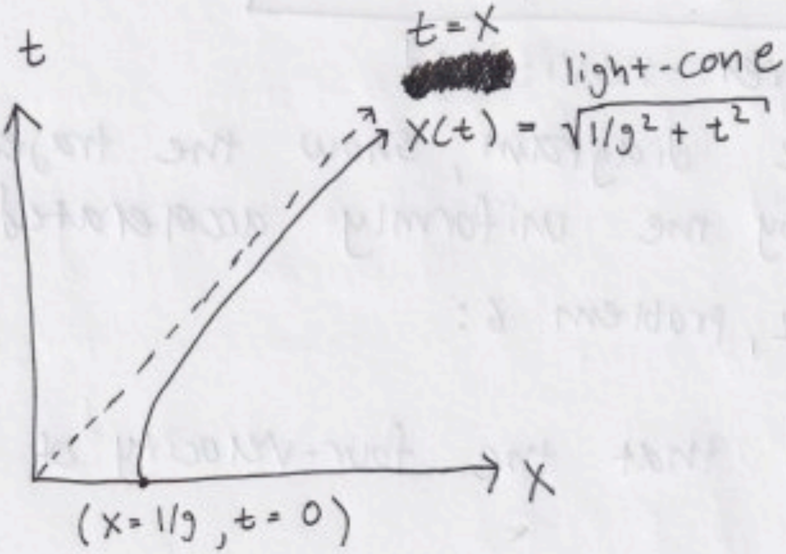
· We can integrate to find the trajectories x + t in terms of $\bar{t}$:

$$t(\bar{t}) = \int_0^{\bar{t}} u^t \, d\bar{t} = \int_0^{\bar{t}} \cosh(g\bar{t}) \, d\bar{t} = \frac{1}{g}\sinh(g\bar{t})$$

$$x(\bar{t}) = \int_0^{\bar{t}} u^x \, d\bar{t} = \int_0^{\bar{t}} \sinh(g\bar{t}) \, d\bar{t} = \frac{1}{g}\cosh(g\bar{t})$$

$\rightarrow \quad x^2 - t^2 = \frac{1}{g^2}$ · Graphing this on t v.s. x we get:

Hyperbolas



$\boxed{b}$ · Show that there is a region that is causally disconnected from the astronaut & events in this region cannot affect the astronaut since <u>information can't travel faster than "c"</u>:

← light cones in here would need to be more shallow ($c > 1$) in order for them to intersect the world line of the astronaut ⋇

$\boxed{c}$ · Find the boundary between the causally connected v.s. disconnected regions of space time. This is called the particle horizon:

· In this region, standard light cones can intersect the world-line of the accelerating observer...

$\boxed{2}$ · In class, I listed one of the defining characteristics of a perfect fluid that it have no viscosity - i.e. no force parallel to the interface between fluid elements. This implied that the stress-energy tensor must be diagonal. I then claimed

$$T^{\alpha\beta} \doteq \mathrm{diag}\{\rho, P, P, P\}$$

in (t, x, y, z) coords. Suppose the form were instead:

$$T^{\alpha\beta} \doteq \mathrm{diag}\{\rho, P(1+\varepsilon), P, P\}$$

Show that performing a rotation about the z-axis by an angle ϕ that $T^{\alpha'\beta'}$ has off-diagonal components of order εP. Hence, we must have $\varepsilon = 0$ in order for the tensor to be diagonal in all Cartesian coordinate definitions:

· Tensors transform like so:

$$T^{\alpha'\beta'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\beta'}}{\partial x^{\beta}}\, T^{\alpha\beta}$$

$\Updownarrow$ $\quad T' = R T R^{-1}$ in matrix notation

$$T' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{V} & 0 & 0 & 0 \\ 0 & \rho(1+\varepsilon) & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{V} & 0 & 0 & 0 \\ 0 & \rho(1+\varepsilon)\cos\theta & \rho(1+\varepsilon)\sin\theta & 0 \\ 0 & -\rho\sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 0 & \rho \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} \mathcal{V} & 0 & 0 & 0 \\ 0 & \rho(1+\varepsilon)\cos^2\theta - \rho\sin^2\theta & \rho\varepsilon\sin\theta\cos\theta & 0 \\ 0 & \rho\varepsilon\sin\theta\cos\theta & \rho(1+\varepsilon\sin^2\theta) & 0 \\ 0 & 0 & 0 & \rho \end{bmatrix} = T'}$$

$\longrightarrow$ · In order for $\overline{\overline{T}}$ to be diagonal in all Cartesian coordinate frames, must have that

$\boxed{\varepsilon \to 0}$

$\boxed{3}$ · An observer with 4-velocity $\vec{U}$ interacting with an e-M field $\vec{F}$ measures electric + magnetic fields $\vec{E}_{\vec{U}}$ and $\vec{B}_{\vec{U}}$ in their instantaneous local inertial reference frame (that is, in an orthonormal basis with $\vec{e}_t = \vec{U}$). These fields are 4-vectors w/ components:

$$E^{\alpha}_{\vec{U}} = F^{\alpha\beta} U_{\beta} \quad ; \quad B^{\alpha}_{\vec{U}} = -\frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} U_{\beta} F_{\gamma\delta}$$

$\boxed{a}$ · Show that $\vec{E}_{\vec{U}}$ and $\vec{B}_{\vec{U}}$ lie orthogonal to the observer's worldline. Thus they are spatial vectors according to the observer, living entirely in that observer's hypersurface of simultaneity:

Recall the projection operator $P^{\alpha\beta} = \eta^{\alpha\beta} + U^{\alpha} U^{\beta}$

$V_{\perp}^{\alpha} \equiv P^{\alpha}_{\ \beta} V^{\beta}$ for an arbitrary $\vec{V}$, and

$$V_{\perp}^{\alpha} U_{\alpha} = 0 \quad (*)$$

· Also $V_{\perp\perp}^{\alpha} = P^{\alpha}_{\ \beta} V_{\perp}^{\beta} = V_{\perp}^{\alpha}$ $(\star)$

· Thus if we can show that $\bigstar$ holds for $\vec{E}_{\vec{v}}$ and $\vec{B}_{\vec{v}}$ (the application of the projection operator leaves the vectors unchanged) then this implies $\circledast$ (that $\vec{E}_{\vec{u}} \cdot \vec{u} = \vec{B}_{\vec{u}} \cdot \vec{u} = 0$):

$$E_{\perp}^{\gamma} \equiv P^{\gamma}{}_{\alpha} E^{\alpha} = P^{\gamma}{}_{\alpha} F^{\alpha\beta} U_{\beta}$$

$$= (\eta^{\gamma}{}_{\alpha} + U^{\gamma} U_{\alpha})(F^{\alpha\beta} U_{\beta})$$

$$= F^{\gamma\beta} U_{\beta} + U^{\gamma} \boxed{U_{\alpha} U_{\beta}} F^{\alpha\beta}$$

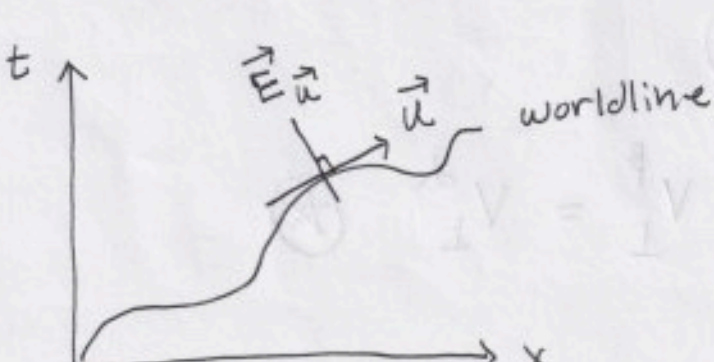
symmetric: $U_{\alpha} U_{\beta} = U_{\beta} U_{\alpha}$

anti-symmetric s.t. $F^{\alpha\beta} = -F^{\beta\alpha}$

Anti-sym $\otimes$ sym $\rightarrow \emptyset$

$$\Rightarrow E_{\perp}^{\gamma} = F^{\gamma\beta} U_{\beta} = E^{\gamma}$$

$\Rightarrow E_{\perp}^{\gamma} = E^{\gamma}$ so the projection operator leaves $\vec{E}_{\vec{v}}$ unchanged implying $\vec{E}_{\vec{v}} \cdot \vec{u} = 0$



· This implies $E^{\alpha}_{\vec{u}}$ is always instantaneously perpendicular to the observer's worldline

QED

· <u>Now</u> we do the same for $B^{\alpha}_{\vec{u}} = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}U_{\beta}F_{\gamma\delta}$

$$B_{\perp}^{\omega} \equiv P^{\omega}_{\ \alpha} B^{\alpha} = P^{\omega}_{\ \alpha}\left(-\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}U_{\beta}F_{\gamma\delta}\right)$$

$$= \left(\eta^{\omega}_{\ \alpha} + U^{\omega}U_{\alpha}\right)\left(-\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}U_{\beta}F_{\gamma\delta}\right)$$

$$= -\frac{1}{2}\varepsilon^{\omega\beta\gamma\delta}U_{\beta}F_{\gamma\delta} - \frac{1}{2}U^{\omega}F_{\gamma\delta}\underbrace{\underset{\substack{\uparrow \\ \text{sym under} \\ \alpha \leftrightarrow \beta}}{U_{\alpha}U_{\beta}}\ \underset{\substack{\uparrow \\ \text{anti-sym under} \\ \alpha \leftrightarrow \beta}}{\varepsilon^{\alpha\beta\gamma\delta}}}_{=\,0}$$

$\Rightarrow B_{\perp}^{\omega} = B^{\omega}$ so this implies that $\vec{B}_{\vec{u}} \cdot \vec{u} = 0$ also. Which itself implies that $B^{\alpha}_{\vec{u}}$ is always instantaneously orthogonal to the observer's worldline. W

$\boxed{b}$. Show that the field tensor can be reconstructed from the observer's 4-velocity and the E & B fields they measure via the following tensor equation (valid for any basis):

$$\boxed{LHS = F^{\alpha\beta} = RHS = U^{\alpha} E^{\beta}_{\vec{u}} - E^{\alpha}_{\vec{u}} U^{\beta} + \varepsilon^{\alpha\beta}{}_{\gamma\delta} U^{\gamma} B^{\delta}_{\vec{u}}}$$

· <u>Show</u> that LHS = RHS:

$$RHS = U^{\alpha} F^{\beta\omega} U_{\omega} - F^{\alpha\beta} \underbrace{U_{\beta} U^{\beta}}_{-1} + \varepsilon^{\alpha\beta}{}_{\gamma\delta} U^{\gamma} \left(-\tfrac{1}{2}\right) \varepsilon^{\delta\beta\gamma\omega} U_{\beta} F_{\gamma\omega}$$

$$= U^{\alpha} \underset{\substack{\uparrow \\ \text{symm. under} \\ \omega \leftrightarrow \beta}}{\eta_{\omega\beta}} U^{\beta} \underset{\substack{\uparrow \\ \text{anti-symm. under} \\ \omega \leftrightarrow \beta}}{F^{\beta\omega}} + F^{\alpha\beta} - \varepsilon^{\alpha}{}_{\beta\gamma\delta} \left(-\tfrac{1}{2}\right) \varepsilon^{\delta\beta\gamma\omega} U^{\gamma} U_{\beta} F_{\gamma\omega}$$

$$= \emptyset + F^{\alpha\beta} - \frac{1}{2} \underbrace{\varepsilon^{\delta\beta\gamma\omega} \varepsilon_{\alpha\beta\gamma\delta}} U^{\gamma} U_{\beta} F_{\gamma\omega}$$

$$\Rightarrow RHS = F^{\alpha\beta} \checkmark\checkmark$$

$$\underbrace{\varepsilon^{\delta\omega\beta\gamma} \varepsilon_{\alpha\delta\beta\gamma}}$$

Identity given in PSET statement

$$\underbrace{2\left(\delta^{\delta}_{\delta} \delta^{\omega}_{\alpha} - \delta^{\delta}_{\alpha} \delta^{\omega}_{\delta}\right)}$$

$$2\left(\delta^{\omega}_{\alpha} - \delta^{\omega}_{\alpha}\right) = \emptyset$$

· So the boxed equation holds true. QED.

$\boxed{c}$ · Show that you can write $\overline{\overline{F}}$ in terms of the wedge product + Hodge dual of $\vec{J}$, $\vec{E}_{\vec{u}}$, and $\vec{B}_{\vec{u}}$

· Wedge product definition:

$$\vec{A} \wedge \vec{B} = \vec{A} \otimes \vec{B} - \vec{B} \otimes \vec{A}$$

$$\Rightarrow \vec{u} \wedge \vec{E}_{\vec{u}} = u^{\alpha} E^{\beta}_{\vec{u}} - E^{\alpha}_{\vec{u}} u^{\beta} \quad (i)$$

· Hodge dual of (0 2) tensor definition:

$$*C_{\nu\nu} = \frac{1}{2} \varepsilon^{\alpha\beta}{}_{\nu\nu} C_{\alpha\beta}$$

$$\rightarrow *C^{\alpha\beta} = \frac{1}{2} \varepsilon_{\gamma\delta}{}^{\alpha\beta} C^{\gamma\delta}$$

$$\rightarrow *C^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta}{}_{\gamma\delta} C^{\gamma\delta}$$

· <u>Also</u> $\vec{u} \wedge \vec{B}_{\vec{u}} = u^{\gamma} B^{\delta}_{\vec{u}} - B^{\gamma}_{\vec{u}} u^{\delta}$

$$\rightarrow *(\vec{u} \wedge \vec{B}_{\vec{u}}) = \frac{1}{2} \varepsilon^{\alpha\beta}{}_{\gamma\delta} (u^{\gamma} B^{\delta}_{\vec{u}} - B^{\gamma}_{\vec{u}} u^{\delta})$$

· Focus on the second term: $\rightsquigarrow$

· Let $\gamma \leftrightarrow \delta$, then second term becomes:

$$-\frac{1}{2} \varepsilon^{\alpha\beta}{}_{\delta\gamma} \, U^{\gamma} B^{\delta}_{\vec{u}}$$

· But $\varepsilon^{\alpha\beta}{}_{\delta\gamma} = -\varepsilon^{\alpha\beta}{}_{\gamma\delta}$

$$\rightarrow \; *(\vec{u} \wedge \vec{B}_{\vec{u}}) = \varepsilon^{\alpha\beta}{}_{\gamma\delta} \, U^{\gamma} B^{\delta}_{\vec{u}} \quad \text{(ii)}$$

· <u>Now</u> (i) + (ii) imply:

$$F^{\alpha\beta} = U^{\alpha} E^{\beta}_{\vec{u}} - E^{\alpha}_{\vec{u}} U^{\beta} + \varepsilon^{\alpha\beta}{}_{\gamma\delta} \, U^{\gamma} B^{\delta}_{\vec{u}}$$

$$= \text{(i)} + \text{(ii)}$$

$$\rightarrow \; \boxed{\overline{\overline{F}} = \vec{u} \wedge \vec{E}_{\vec{u}} + *(\vec{u} \wedge \vec{B}_{\vec{u}})} \quad \text{QED}$$

$\boxed{4}$ $\boxed{a}$ · Show that under a coordinate transformation, the components of the Christoffel symbol transform as follows:

$$\Gamma^{\alpha'}_{\beta'\gamma'} = \underbrace{\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \Gamma^{\alpha}_{\beta\gamma}}_{\text{Tensorial}} \pm \underbrace{\frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\gamma}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}}_{\text{NOT Tensorial}}$$

$$\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\nu}\Gamma_{\nu\beta\gamma} \quad \leftarrow \quad \text{By } \underline{\underline{\text{definitions}}}$$

$$= \frac{1}{2} g^{\alpha\nu}\left(\partial_\beta g_{\gamma\nu} + \partial_\gamma g_{\nu\beta} - \partial_\nu g_{\beta\gamma}\right)$$

- The metric transforms like:

$$g_{\alpha'\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} g_{\alpha\beta}$$

- Partials transform like:

$$\partial_{\alpha'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \partial_\alpha$$

- So overall, we get that the Christoffel transforms as:

$$\Gamma^{\alpha'}_{\beta'\gamma'} = \left(\frac{1}{2} \cdot \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} g^{\alpha\nu}\right)\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}} \partial_\beta\left(\frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \cdot \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\gamma\nu}\right)\right.$$

$$\left. + \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \partial_\gamma\left(\frac{\partial x^{\nu}}{\partial x^{\nu'}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} g_{\nu\beta}\right) - \frac{\partial x^{\nu}}{\partial x^{\nu'}} \partial_\nu\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} g_{\beta\gamma}\right)\right)$$

$$= \left(\frac{1}{2} \cdot \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \cancel{\frac{\partial x^{\nu'}}{\partial x^{\nu}}} g^{\alpha\nu}\right)\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \cdot \cancel{\frac{\partial x^{\nu}}{\partial x^{\nu'}}} \cdot \partial_\beta g_{\gamma\nu}\right.$$

$$\left. + \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \cdot \cancel{\frac{\partial x^{\nu}}{\partial x^{\nu'}}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \partial_\gamma g_{\nu\beta} - \cancel{\frac{\partial x^{\nu}}{\partial x^{\nu'}}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \partial_\nu g_{\beta\gamma}\right)$$

$+$ ★ $\leftarrow$ defined on next page

$$= \underbrace{\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \Gamma^{\alpha}_{\beta\gamma}}_{\text{Tensorial}} + ✪$$

Where ✪ $\equiv \left(\frac{1}{2} \cdot \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} g^{\alpha\nu} \right)$

$$\cdot \Bigg(\frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \cdot \frac{\partial^2 x^{\nu}}{\partial x^{\beta} \partial x^{\nu'}} g_{\gamma\nu}$$

$$+ \cancel{\frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial^2 x^{\gamma}}{\partial x^{\beta} \partial x^{\gamma'}} \cdot \frac{\partial x^{\nu}}{\partial x^{\nu'}}} g_{\gamma\nu} \leftarrow \text{redefine } g_{\gamma\nu} = \delta^{\beta}_{\nu} g_{\beta\gamma}$$

$$+ \frac{\partial x^{\nu}}{\partial x^{\nu'}} \cdot \cancel{\frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \cdot \frac{\partial^2 x^{\beta}}{\partial x^{\gamma} \partial x^{\beta'}}} g_{\nu\beta} \leftarrow \text{redefine } g_{\nu\beta} = \delta^{\gamma}_{\nu} g_{\beta\gamma}$$

$$+ \frac{\partial^2 x^{\nu}}{\partial x^{\gamma} \partial x^{\nu'}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} g_{\nu\beta}$$

$$- \cancel{\frac{\partial x^{\nu}}{\partial x^{\nu'}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial^2 x^{\gamma}}{\partial x^{\nu} \partial x^{\gamma'}}} g_{\beta\gamma}$$

$$- \frac{\partial x^{\nu}}{\partial x^{\nu'}} \cdot \cancel{\frac{\partial^2 x^{\beta}}{\partial x^{\nu} \partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}} g_{\beta\gamma} \Bigg)$$

$\Rightarrow$ 3rd + 6th term cancel and 2nd + 5th terms cancel after applying our Kronecker Delta tricks W

$$\Rightarrow \circledast = \left(\frac{1}{2} \cdot \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} g^{\alpha\nu}\right)\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\right)$$

$$\cdot \underbrace{\left(\frac{\partial^2 x^{\nu}\, g_{\gamma\nu}}{\partial x^{\beta} \partial x^{\nu'}} + \frac{\partial^2 x^{\nu}\, g_{\nu\beta}}{\partial x^{\gamma} \partial x^{\nu'}}\right)}$$

$$-\frac{\partial^2 x^{\nu}}{\partial x^{\beta} \partial x^{\nu'}} \delta^{\alpha}_{\gamma}\, g_{\alpha\nu} - \frac{\partial^2 x^{\nu}}{\partial x^{\gamma} \partial x^{\nu'}} \delta^{\alpha}_{\beta}\, g_{\alpha\nu}$$

(see subsequent proof)
minus sign comes from partial derivative tricks

$$\Rightarrow (A) = -\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\right)\left(\frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\gamma}}\right)$$

<u>overall</u> ↴

$$\boxed{\Rightarrow \Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \Gamma^{\alpha}_{\beta\gamma} - \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\gamma}}\left(\frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\right)}$$

$\boxed{b}$ · Using this rule, show that:

$$\nabla_{\alpha'} A^{\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \frac{\partial x^{\beta'}}{\partial x^{\beta}} \nabla_{\alpha} A^{\beta}$$ like a tensor as expected:

$$\nabla_{\alpha'} A^{\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \partial_{\alpha}\left(\frac{\partial x^{\beta'}}{\partial x^{\beta}} A^{\beta}\right) + \Gamma^{\beta'}_{\nu'\alpha'} A^{\nu'}$$

$$= \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \frac{\partial x^{\beta'}}{\partial x^{\beta}} \partial_{\alpha} A^{\beta} + \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} A^{\beta} \frac{\partial^2 x^{\beta'}}{\partial x^{\alpha} \partial x^{\beta}}$$

$$+ \left(\left(\frac{\partial x^{\beta'}}{\partial x^{\beta}} \cdot \cancel{\frac{\partial x^{\nu}}{\partial x^{\nu'}}} \cdot \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \Gamma^{\beta}_{\nu\alpha} - \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \cancel{\frac{\partial x^{\nu}}{\partial x^{\nu'}}} \cdot \frac{\partial^2 x^{\beta'}}{\partial x^{\nu} \partial x^{\alpha}} \right) \cancel{\frac{\partial x^{\nu'}}{\partial x^{\nu}}} A^{\nu} \right.$$

$$= \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \frac{\partial x^{\beta'}}{\partial x^{\beta}} \partial_{\alpha} A^{\beta} + \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \frac{\partial x^{\beta'}}{\partial x^{\beta}} \Gamma^{\beta}_{\nu\alpha} A^{\nu}$$

$$+ \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \cancel{\frac{\partial^2 x^{\beta'}}{\partial x^{\alpha} \partial x^{\beta}}} A^{\beta} - \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \cancel{\frac{\partial^2 x^{\beta'}}{\partial x^{\alpha} \partial x^{\nu}}} A^{\nu}$$

dummy indices

$$\rightarrow \boxed{\nabla_{\alpha'} A^{\beta'}} = \left(\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \frac{\partial x^{\beta'}}{\partial x^{\beta}} \right) \left(\partial_{\alpha} A^{\beta} + \Gamma^{\beta}_{\nu\alpha} A^{\nu} \right)$$

$$= \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \cdot \frac{\partial x^{\beta'}}{\partial x^{\beta}} \nabla_{\alpha} A^{\beta} \quad ✓$$

which is indeed tensorial!

$\boxed{\text{Proof of minus sign in part 4.a:}}$

given $\frac{1}{2} \cdot \underbrace{\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} \cdot \frac{\partial^2 x^{\nu}}{\partial x^{\beta} \partial x^{\nu'}}} \delta^{\alpha}_{\gamma} \underbrace{g^{\alpha\nu} g_{\alpha\nu}}_{1} \equiv *$

$$\underbrace{\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}}_{\equiv g} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} \cdot \frac{\partial}{\partial \beta}\underbrace{\left(\frac{\partial x^{\nu}}{\partial x^{\nu'}} \right)}_{\equiv f} \equiv (i)$$

$$\rightarrow g' = \frac{\partial^2 x^{\alpha'}}{\partial x^{\alpha} \partial x^{\beta}}$$

$$\rightarrow (i) = \left(\frac{\partial x^{\nu'}}{\partial x^{\nu}} \right) \left((fg)' - g'f \right)$$

$$(fg)' = \left(\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\nu}}{\partial x^{\nu'}} \right)' =$$ derivative of forward times backward transformation

$$= \partial_{\beta} (\text{Identity}) = \emptyset$$

$$\rightarrow (i) = -\frac{\partial x^{\nu'}}{\partial x^{\nu}} \cdot \frac{\partial x^{\nu}}{\partial x^{\nu'}} \cdot \frac{\partial^2 x^{\alpha'}}{\partial x^{\alpha} \partial x^{\beta}}$$

$$\rightarrow * = -\frac{1}{2} \cdot \frac{\partial^2 x^{\alpha'}}{\partial x^{\gamma} \partial x^{\beta}}$$ and similarly for other term, hence the minus sign...

$\boxed{5}$ · <u>Carroll</u>, Chapter 3, Problem 3.

[redacted]

· Imagine we have a diagonal metric $g_{\mu\nu}$. Show that the Christoffel symbols are given by:

(i) $\Gamma^{\lambda}_{\mu\nu} = 0$

(ii) $\Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu}$

(iii) $\Gamma^{\lambda}_{\mu\lambda} = \partial_\mu\left(\ln\sqrt{|g_{\lambda\lambda}|}\right)$

(iv) $\Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda\left(\ln\sqrt{|g_{\lambda\lambda}|}\right)$

· <u>for (i)</u>: $\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\alpha}\Gamma_{\alpha\mu\nu}$

$$= \frac{1}{2}g^{\lambda\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)$$

• We will also need a few other assumptions:

① $\partial_\alpha g_{\beta\gamma} = 0$ ("metric compatibility"... True in Cartesian representation, so true in general)

$\quad\ \ \| $

$\nabla_\alpha g_{\beta\gamma}$

definition of covariant derivative of metric

② $\nabla_\beta g_{\nu r} = \partial_\beta g_{\nu r} - \Gamma^\alpha_{\beta\nu} g_{\alpha r} - \Gamma^\alpha_{\beta r} g_{\nu\alpha}$

$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ symmetry of Christoffel

• Since the last term in ⓘ is off-diagonal, it's zero, so we get:

$$\Gamma^\lambda_{\nu r} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_\nu g_{r\alpha} + \partial_r g_{\alpha\nu} \right)$$

• expand + only consider on-diagonal terms:

$$= \frac{1}{2} g^{\lambda r} \partial_\nu g_{rr} + \frac{1}{2} g^{\lambda\nu} \partial_r g_{\nu\nu} \quad (*)$$

<u>Aside</u> From ① + ② we get:

$$0 = \partial_\nu g_{\lambda r} - \Gamma^\alpha_{\nu\lambda} g_{\alpha r} - \Gamma^\alpha_{\nu r} g_{\lambda\alpha}$$

$$\rightarrow \quad \partial_\nu g_{rr} = \Gamma^\alpha_{\nu r} g_{\alpha r} + \Gamma^\alpha_{\nu r} g_{\alpha r}$$

$$\partial_\nu g_{rr} = 2\Gamma^r_{\nu r} g_{rr} \quad ⓐ$$

· Also

$$\partial_\sigma g_{\mu\mu} = 2\Gamma^{\mu}_{\sigma\mu}\, g_{\mu\mu} \quad (b)$$

· plugging (a) + (b) into (*) we get:

$$\Gamma^{\lambda}_{\mu\sigma} = \tfrac{1}{\cancel{2}} g^{\lambda\sigma} \cdot \cancel{2}\Gamma^{\sigma}_{\mu\sigma}\, g_{\sigma\sigma} + \tfrac{1}{\cancel{2}} g^{\lambda\mu} \cdot \cancel{2}\Gamma^{\mu}_{\sigma\mu}\, g_{\mu\mu}$$

$$= \delta^{\lambda}_{\sigma}\,\Gamma^{\sigma}_{\mu\sigma} + \delta^{\lambda}_{\mu}\,\Gamma^{\mu}_{\sigma\mu}$$

$$= \Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\mu\sigma}$$

$$= 2\Gamma^{\lambda}_{\mu\sigma}$$

$$\rightarrow \boxed{\Gamma^{\lambda}_{\mu\sigma} = 0}$$

· Now prove (ii):

$$\Gamma^{\lambda}_{\mu\mu} = \tfrac{1}{2} g^{\lambda\alpha}\left(\partial_\mu g_{\mu\alpha} + \partial_\mu g_{\mu\alpha} - \partial_\alpha g_{\mu\mu}\right)$$

$$= g^{\lambda\alpha}\,\partial_\mu g_{\mu\alpha} - \tfrac{1}{2} g^{\lambda\alpha}\,\partial_\alpha g_{\mu\mu}$$

$$= g^{\lambda\lambda}\,\underbrace{\cancel{\partial_\mu g_{\mu\lambda}}}_{0} - \tfrac{1}{2} g^{\lambda\lambda}\,\partial_\lambda g_{\mu\mu} \rightsquigarrow$$

$$(g_{\lambda\lambda})^{-1} \equiv g^{\lambda\lambda}$$

$$\rightarrow \boxed{\Gamma^{\lambda}_{\nu\nu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\nu\nu}}$$

· Now prove (iii):

$$\Gamma^{\lambda}_{\nu\lambda} = \frac{1}{2} g^{\lambda\alpha}\left(\partial_\nu g_{\lambda\alpha} + \cancelto{0}{\partial_\lambda g_{\nu\alpha}} - \cancelto{0}{\partial_\alpha g_{\nu\lambda}}\right)$$

$$= \frac{1}{2} g^{\lambda\lambda}\partial_\nu g_{\lambda\lambda} = \text{LHS}$$

· Now consider $\partial_\nu \ln\sqrt{|g_{\lambda\lambda}|}$ via chain rule:

$$= \frac{1}{\sqrt{|g_{\lambda\lambda}|}} \cdot \frac{1}{2}|g_{\lambda\lambda}|^{-1/2} \cdot \partial_\nu |g_{\lambda\lambda}|$$

$$= \frac{\partial_\nu |g_{\lambda\lambda}|}{2|g_{\lambda\lambda}|}$$

. since we are dividing $g_{\lambda\lambda}$ by itself we can take away the $|\,|$'s (absolute values)

$$= \frac{1}{2} g^{\lambda\lambda}\partial_\nu g_{\lambda\lambda} \equiv \text{RHS} = \text{LHS}$$

using $1/g_{\lambda\lambda} = g^{\lambda\lambda}$ $\rightarrow \boxed{\Gamma^{\lambda}_{\lambda\nu} = \partial_\nu \ln\sqrt{|g_{\lambda\lambda}|}}$

· Now prove (iv):

· Replacing $\nu \to \lambda$ in the last equation, we get that:

$$\boxed{\Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda \left(\ln \sqrt{|g_{\lambda\lambda}|} \right)}$$

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$\boxed{6}$. <u>Carroll</u>, Chapter 3, Problem 2:

- For an arbitrary vector V^α in 3-space, use the covariant derivative ∇_α in spherical polar coords to find the gradient of a scalar $\bar{\nabla}\phi$, the divergence of the vector V^α, and the curl $\bar{\nabla}\times\bar{V}$:

$$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta$$

- We will ultimately end up needing a bunch of Christoffel symbols which we will calculate by taking partials of our 3-space metric
- We therefore need to find the metric of our 3-space to begin with:

$$ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

$$\Rightarrow g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

- and the inverse of the metric is given by

$$g^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{bmatrix}$$

$$\Gamma^{\lambda}_{\nu r} = g^{\lambda\alpha}\Gamma_{\alpha\nu r} \longleftarrow$$ By definitions

$$= \frac{1}{2} g^{\lambda\alpha}\left(\partial_{\nu} g_{r\alpha} + \partial_{r} g_{\alpha\nu} - \partial_{\alpha} g_{\nu r}\right)$$

$$\Rightarrow \Gamma^{r}_{rr} = \frac{1}{2} g^{r\alpha}\left(\partial_{r} g_{r\alpha} + \partial_{r} g_{\alpha r} - \overset{\nearrow 0}{\partial_{\alpha} g_{rr}}\right)$$

$$= g^{r\alpha}\partial_{r} g_{r\alpha} = [1\ 0\ 0]\cdot\vec{0} = \emptyset$$

$$\Rightarrow \Gamma^{r}_{r\theta} = \frac{1}{2} g^{r\alpha}\left(\partial_{r} g_{\theta\alpha} + \underset{\searrow 0}{\partial_{\theta} g_{\alpha r}} - \overset{\nearrow 0}{\partial_{\alpha} g_{r\theta}}\right)$$

$$= \frac{1}{2}[1\ 0\ 0]\begin{bmatrix} 0 \\ 2r \\ 0 \end{bmatrix} = \emptyset$$

$$\Rightarrow \Gamma^{r}_{r\varphi} = \frac{1}{2} g^{r\alpha}\left(\partial_{r} g_{\varphi\alpha} + \overset{\nearrow 0}{\partial_{\varphi} g_{\alpha r}} - \partial_{\alpha} g_{r\varphi}\right)$$

$$= \frac{1}{2}[1\ 0\ 0]\left(\begin{bmatrix} 0 \\ 0 \\ 2r\sin^2\theta \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= \emptyset$$

$$\Rightarrow \Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\alpha}\left(\overset{\nearrow 0}{\partial_\theta g_{\theta\alpha}} + \overset{\nearrow 0}{\partial_\theta g_{\alpha\theta}} - \partial_\alpha g_{\theta\theta}\right)$$

$$= -\frac{1}{2}\begin{bmatrix} 0 & r^{-2} & 0 \end{bmatrix}\begin{pmatrix} 2r \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \Gamma^{\theta}_{\theta r} = \frac{1}{2} g^{\theta\alpha}\left(\overset{\nearrow 0}{\partial_\theta g_{r\alpha}} + \partial_r g_{\alpha\theta} - \overset{\nearrow 0}{\partial_\alpha g_{r\theta}}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & r^{-2} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2r \\ 0 \end{bmatrix} = 1/r$$

$$\Rightarrow \Gamma^{\theta}_{\theta\varphi} = \frac{1}{2} g^{\theta\alpha}\left(\partial_\theta g_{\varphi\alpha} + \overset{\nearrow 0}{\partial_\varphi g_{\alpha r}} - \overset{\nearrow 0}{\partial_\alpha g_{r\varphi}}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & r^{-2} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2r^2 \sin\theta\cos\theta \end{bmatrix} = 0$$

$$\Rightarrow \Gamma^{\varphi}_{\varphi\varphi} = \frac{1}{2} g^{\varphi\alpha}\left(\overset{\nearrow 0}{\partial_\varphi g_{\varphi\alpha}} + \overset{\nearrow 0}{\partial_\varphi g_{\alpha\varphi}} - \partial_\alpha g_{\varphi\varphi}\right]$$

$$= -\frac{1}{2}\begin{bmatrix} 0 & 0 & 1/r^2\sin^2\theta \end{bmatrix}\begin{bmatrix} 2r\sin^2\theta \\ 2r^2\sin\theta\cos\theta \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow \Gamma^{\varphi}_{\varphi r} = \frac{1}{2}\, g^{\varphi\alpha}\left(\overset{\nearrow 0}{\partial_{\varphi} g_{r\alpha}} + \partial_r g_{\alpha\varphi} - \overset{\nearrow 0}{\partial_{\alpha} g_{\varphi r}}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 1/r^2\sin^2\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2r\sin^2\theta \end{bmatrix} = 1/r$$

$\Rightarrow$ Finally ☺,

$$\Gamma^{\varphi}_{\varphi\theta} = \frac{1}{2}\, g^{\varphi\alpha}\left(\overset{\nearrow 0}{\partial_{\varphi} g_{\theta\alpha}} + \partial_{\theta} g_{\alpha\varphi} - \overset{\nearrow 0}{\partial_{\alpha} g_{\varphi\theta}}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 1/r^2\sin^2\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2r^2\sin\theta\cos\theta \end{bmatrix} = \cot\theta$$

- So the only non-zero Christoffels so far are:

$$\Gamma^{\varphi}_{\theta\varphi} = \cot\theta, \quad \Gamma^{\varphi}_{\varphi r} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}$$

- Using our christoffel's, lets now calculate

$$\mathrm{div}(v^{\alpha}) = \nabla_{\alpha} v^{\alpha} = \partial_{\alpha} v^{\alpha} + \Gamma^{\alpha}_{\alpha\nu} v^{\nu}$$

$$= \partial_r V_r + \partial_\theta V_\theta + \partial_\varphi V^\varphi$$

$$+ \cancel{\Gamma^r_{rr} V^r} + \cancel{\Gamma^r_{r\theta} V^\theta} + \cancel{\Gamma^r_{r\varphi} V^\varphi}$$

$$+ \Gamma^\theta_{\theta r} V_r + \cancel{\Gamma^\theta_{\theta\theta} V^\theta} + \cancel{\Gamma^\theta_{\theta\varphi} V^\varphi}$$

$$+ \Gamma^\varphi_{\varphi r} V^r + \Gamma^\varphi_{\varphi\theta} V^\theta + \cancel{\Gamma^\varphi_{\varphi\varphi} V^\varphi}$$

$$\boxed{= \partial_r V_r + \partial_\theta V_\theta + \partial_\varphi V^\varphi + \frac{2V_r}{r} + \cot(\theta) V^\theta = \operatorname{div}(V^\alpha)}$$

. Notice that this does not match the standard formula one would expect for the $\overline{\nabla} \cdot \vec{V}$ in Euclidean space. This is because we are working in a coordinate basis rather than an orthonormal basis W

. Now onwards to calculate $\nabla \phi$:

$$\overline{\nabla} \phi \rightarrow \partial_\alpha \phi \, \vec{e}_\alpha$$

$$\boxed{\overline{\nabla} \phi = \partial_r \phi \vec{e}_r + \partial_\theta \phi \vec{e}_\theta + \partial_\varphi \phi \vec{e}_\varphi}$$

· Now for the curl. Take the definition

$$\mathrm{curl}(\vec{V})_i = \varepsilon_i{}^{jk} \nabla_j V_k$$

$$= \varepsilon_i{}^{jk} \left(\partial_j V_k + \Gamma^{\nu}_{jk} V_\nu \right)$$

$$\rightarrow \mathrm{curl}(\vec{V})_r = \varepsilon_r{}^{\theta\varphi} \left(\partial_\theta V_r - \partial_r V_\theta + \left(\Gamma^{\nu}_{r\theta} - \Gamma^{\nu}_{\theta r} \right) V_\nu \right)$$

sym $\Downarrow$ $= \emptyset$

· Define $\varepsilon_r{}^{\theta\varphi} = +1$

$$\rightarrow \mathrm{curl}(\vec{V})_r = \partial_\theta V_r - \partial_r V_\theta$$

$$\rightarrow \mathrm{curl}(\vec{V})_\theta = \varepsilon_\theta{}^{r\varphi} \left(\partial_r V_\varphi - \partial_\varphi V_r \right)$$

$$= -\left(\partial_r V_\varphi - \partial_\varphi V_r \right)$$

$$\rightarrow \mathrm{curl}(\vec{V})_\varphi = \underbrace{\varepsilon_\varphi{}^{r\theta}}_{+1} \left(\partial_r V_\theta - \partial_\theta V_r \right)$$

· <u>So overall;</u>

$$\boxed{\mathrm{curl}(\vec{V}) = \left(\partial_\theta V_r - \partial_r V_\theta \right) \vec{e}_r + \left(\partial_\varphi V_r - \partial_r V_\varphi \right) \vec{e}_\theta + \left(\partial_r V_\theta - \partial_\theta V_r \right) \vec{e}_\varphi}$$

[7] . Prove the following connection identities

[a] $\partial_\lambda g_{\nu\sigma} \overset{?}{=} \Gamma_{\nu\sigma\lambda} + \Gamma_{\sigma\nu\lambda}$

$$\text{RHS} = \tfrac{1}{2}\left(\cancel{\partial_\sigma g_{\nu\lambda}} + \partial_\lambda g_{\nu\sigma} - \cancel{\partial_\nu g_{\sigma\lambda}}\right) + \tfrac{1}{2}\left(\cancel{\partial_\nu g_{\sigma\lambda}} + \partial_\lambda g_{\nu\sigma} - \cancel{\partial_\sigma g_{\nu\lambda}}\right) = \partial_\lambda g_{\nu\sigma} \quad \text{QED}$$

[b] $g_{\nu\kappa}\,\partial_\lambda g^{\kappa\sigma} \overset{?}{=} -g^{\kappa\sigma}\,\partial_\lambda g_{\nu\kappa}$

$$\text{LHS} = g_{\nu\kappa}\,\partial_\lambda g^{\kappa\sigma}$$

$$= g_{\nu\kappa}\left(-\Gamma^{\kappa}_{\lambda\alpha} g^{\alpha\sigma} - \Gamma^{\sigma}_{\lambda\alpha} g^{\alpha\kappa}\right)$$

← see notes for why... comes from $\nabla_\lambda g^{\kappa\sigma}$ formula

relabel dummy $\alpha \to \nu$

$$= -g^{\nu\sigma} g_{\nu\kappa} \Gamma^{\kappa}_{\lambda\nu} - g^{\alpha\kappa} g_{\nu\kappa} \Gamma^{\sigma}_{\lambda\alpha}$$

$$= -\delta^{\sigma}_{\kappa} \Gamma^{\kappa}_{\lambda\nu} - \delta^{\alpha}_{\nu} \Gamma^{\sigma}_{\lambda\alpha}$$

$$= -\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\sigma}_{\lambda\nu} = -2\Gamma^{\sigma}_{\lambda\nu} \quad (1)$$

$$\text{RHS} = -g^{\kappa\sigma}\,\partial_\lambda g_{\nu\kappa}$$

$$= -g^{\kappa\sigma}\left(\Gamma^{\alpha}_{\lambda\nu} g_{\alpha\kappa} + \Gamma^{\alpha}_{\lambda\kappa} g_{\nu\alpha}\right)$$

$$= -g^{\kappa\sigma} g_{\alpha\kappa} \Gamma^{\alpha}_{\lambda\nu} - g^{\kappa\sigma} g_{\nu\sigma} \Gamma^{\sigma}_{\lambda\kappa}$$

$$= -\delta^{\sigma}_{\alpha} \Gamma^{\alpha}_{\lambda\nu} - \delta^{\kappa}_{\nu} \Gamma^{\sigma}_{\lambda\kappa}$$

$$= -2\Gamma^{\sigma}_{\lambda\nu} \quad (2)$$, (1) = (2) $\Rightarrow$ LHS = RHS

$$\Rightarrow \quad g_{\nu\kappa} \partial_\lambda g^{\kappa\sigma} = -g^{\kappa\sigma} \partial_\lambda g_{\nu\kappa}$$ QED

$\boxed{c}$ Prove that $\partial_\lambda g^{\nu\sigma} = -\Gamma^{\nu}_{\lambda\kappa} g^{\kappa\sigma} - \Gamma^{\sigma}_{\lambda\kappa} g^{\kappa\nu}$

. This just follows from a relation we already used to prove (a) + (b):

$$\nabla_\lambda g^{\nu\sigma} = \partial_\lambda g^{\nu\sigma} + \Gamma^{\nu}_{\lambda\kappa} g^{\kappa\sigma} + \Gamma^{\sigma}_{\lambda\kappa} g^{\nu\kappa}$$

Definition of covariant derivative of two-index tensor...

$$\nabla_\lambda g^{\nu\sigma} = 0$$

$$\rightarrow \partial_\lambda g^{\nu\sigma} = -\Gamma^{\nu}_{\lambda\kappa} g^{\kappa\sigma} - \Gamma^{\sigma}_{\lambda\kappa} g^{\kappa\nu}$$ QED !!

$\boxed{d}$ · Define $g = \det(g_{\nu\sigma})$

· Prove that $\nabla_\sigma A^{\sigma}_{\nu} = |g|^{-1/2} \partial_\sigma \left(|g|^{1/2} A^{\sigma}_{\nu}\right) - \Gamma^{\lambda}_{\sigma\nu} A^{\sigma}_{\lambda}$

in a coordinate basis:

$\rightsquigarrow$

- Use the fact that $\Gamma^{\sigma}_{\sigma\alpha} = \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}\right)$

- expand LHS $= \nabla_\sigma A^\sigma_\nu$

$$= \partial_\sigma A^\sigma_\nu + \Gamma^\sigma_{\sigma\alpha} A^\alpha_\nu - \Gamma^\alpha_{\sigma\nu} A^\sigma_\alpha$$

"+" for top index part of tensor

"−" for lower index part of tensor

$$= \partial_\sigma A^\sigma_\nu + \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}\right) A^\alpha_\nu - \Gamma^\alpha_{\sigma\nu} A^\sigma_\alpha$$

relabel $\alpha \to \sigma$

relabel $\alpha \to \lambda$

$$= \boxed{\frac{\partial_\sigma\left(\sqrt{|g|}A^\sigma_\nu\right)}{\sqrt{|g|}} - \Gamma^\lambda_{\sigma\nu} A^\sigma_\lambda = \nabla_\sigma A^\sigma_\nu \quad ✓}$$

e · Show that $\nabla_\sigma F^{\nu\sigma} = \sqrt{|g|}\,\partial_\sigma\left(\sqrt{|g|}F^{\nu\sigma}\right)$

in a coordinate basis if $F^{\nu\sigma} = -F^{\sigma\nu}$

(anti-symmetric):

· By definition:

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} + \Gamma^\mu_{\nu\alpha} F^{\alpha\nu} + \Gamma^\nu_{\nu\alpha} F^{\mu\alpha}$$

$$= \partial_\nu F^{\mu\nu} + \Gamma^\mu_{\nu\alpha} F^{\alpha\nu} + \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|}) F^{\mu\alpha}$$

$$= \frac{\partial_\nu (\sqrt{|g|} F^{\mu\nu})}{\sqrt{|g|}} + \underbrace{\Gamma^\mu_{\nu\alpha} F^{\alpha\nu}}_{=0}$$

anti-sym. under $\alpha \longleftrightarrow \nu$

sym. under $\alpha \longleftrightarrow \nu$

$$\Downarrow$$

$$\boxed{\nabla_\nu F^{\mu\nu} = \frac{\partial_\nu (\sqrt{|g|} F^{\mu\nu})}{\sqrt{|g|}}}$$

$\boxed{f}$ · prove that $\Box S \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu S$

$$= \frac{\partial_\mu (\sqrt{|g|}\, g^{\mu\nu} \partial_\nu S)}{\sqrt{|g|}}$$

· In a coordinate basis where S is a scalar function:

$$\text{LHS} = \Box S \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu S = g^{\mu\nu} \nabla_\mu (\partial_\nu S)$$

1-form

$$= g^{\nu\sigma}\partial_\nu\partial_\sigma S - g^{\nu\sigma}\Gamma^{\kappa}_{\nu\sigma}\partial_\kappa S$$

· Now the RHS:

$$\text{RHS} = \frac{\partial_\nu(\sqrt{|g|}\, g^{\nu\sigma}\partial_\sigma S)}{\sqrt{|g|}}$$

$$= \underbrace{(\partial_\sigma S)(\partial_\nu g^{\nu\sigma})}_{\parallel} + g^{\nu\sigma}\partial_\nu\partial_\sigma S + (\partial_\sigma S)(g^{\nu\sigma})\frac{\partial_\nu(\sqrt{|g|})}{\sqrt{|g|}}$$

$$\underbrace{(\partial_\sigma S)\left(-\Gamma^{\nu}_{\nu\kappa}g^{\kappa\sigma} - \Gamma^{\sigma}_{\nu\kappa}g^{\nu\kappa}\right)}_{\parallel}$$

$$(\partial_\sigma S)\left(-g^{\kappa\sigma}\frac{\partial_\kappa(\sqrt{|g|})}{\sqrt{|g|}} - \Gamma^{\sigma}_{\nu\kappa}g^{\nu\kappa}\right)$$

reindex $\kappa \to \nu$ + cancels here.

$$\Rightarrow \text{RHS} = g^{\nu\sigma}\partial_\nu\partial_\sigma S - \partial_\sigma S\,\Gamma^{\sigma}_{\nu\kappa}g^{\nu\kappa}$$

Let $\kappa \leftrightarrow \sigma$

$$\text{RHS} = g^{\nu\sigma}\partial_\nu\partial_\sigma S - g^{\nu\sigma}\Gamma^{\kappa}_{\nu\sigma}\partial_\kappa S = \text{LHS} \quad ✓✓$$

Q.E.D. :)